Defining Reactive Power in Circuit Transients via Local Orthonormal Representations

H. Lev-Ari A.M. Stanković

Abstract— **The paper introduces the notion of reactive power during circuit transients. The definition we propose is based on the concept of local Hilbert space. It reduces to well known quantities in steady state, and it is applicable to systems with arbitrary number of phases. We present the concept in terms of two local representations, namely Short-Time Fourier Coefficients and Haar Wavelets. We illustrate this "dynamic" reactive power on a simple example of a linear RL circuit for which closed-form expressions can be derived.**

I. INTRODUCTION

In many energy processing applications there has been recently an increased interest in concepts of reactive (inactive) power in cases of multiple driving frequencies and in transients. In large part this is in response to the emergence of new classes of loads that are interfaced with utility systems through power electronic converters. In particular, literature on electric drives and power electronic converters focuses on the so called instantaneous reactive power [1] which is obtained via an instantaneous projection of the current onto the voltage. This quantity is identically zero in single phase systems, and turns out to be related the part of reactive power that can be compensated without energy storage [2]. Another recent suggestion from the literature is to use the "ac part" of the instantaneous power [3]. This signal is not very indicative of loads commonly perceived as reactive, as it is nonzero in steady state even for a linear resistor.

In this paper we follow the path outlined in [2] and [4] to define reactive power in transients. The definitions we propose are applicable to an arbitrary number of phases. We illustrate our development on a single phase system because it suffices to demonstrate salient features of the problem, while the necessary notation remains simple. It will turn out that the concept of a local Hilbert space is instrumental for our development, as we explain shortly.

The authors are with the Department of Electrical and Computer Engineering, Northeastern University, Boston. E-mails:{levari,astankov}@ece.neu.edu

II. TIME-VARIANT REAL AND REACTIVE POWER

Consider an n-phase system, i.e., a system with $n+1$ conductors ("wires") in which first n are referenced either to a common ground, or the $(n+1)$-st ("neutral") conductor. Then we can define the n-dimensional voltage $v(t)$ and current vectors $i(t)$, where all currents have reference directions "toward" the load. The instantaneous power delivered to the load is defined as $p(t) \stackrel{\text{def}}{=} i(t)^\top v(t)$. We define the time-variant real (active) power as a "short-term" DC component (local in time) of the product $p(t)$, namely

$$P_0(t) \stackrel{\text{def}}{=} \frac{1}{T}\int_{t-T}^{t} i(s)^\top v(s)ds \tag{1}$$

Note that this quantity need not be positive *for every* t, but its long term average is positive for passive loads. This result motivates us to introduce a time-variant Hilbert space framework for real-valued polyphase signals with the finite local power, viz.

$$\frac{1}{T}\int_{t-T}^{t} \|x(s)\|^2 ds < \infty, \quad \text{for all } t$$

For a fixed "t" our space consists of finite signal segments, and we define the (time-variant) inner product as

$$\langle\, x\,,\, y\,\rangle(t) \stackrel{\text{def}}{=} \frac{1}{T}\int_{t-T}^{t} y(s)^\top x(s)ds \tag{2}$$

With this definition, $P_0(t) = \langle\, v\,,\, i\,\rangle(t)$.

Similarly, the time-variant rms voltage and current are

$$\|v\|(t) \stackrel{\text{def}}{=} \sqrt{\langle\, v\,,\, v\,\rangle(t)},$$

$$\|i\|(t) \stackrel{\text{def}}{=} \sqrt{\langle\, i\,,\, i\,\rangle(t)}$$

Because of the Cauchy-Schwarz inequality we have $P_0^2(t) \leq \|v\|^2(t)\|i\|^2(t)$, which suggests the definition of time-varying reactive power

$$Q(t) \stackrel{\text{def}}{=} \sqrt{\|v\|^2(t)\ \cdot\ \|i\|^2(t) - P_0^2(t)} \tag{3}$$

and the time-varying power factor

$$\cos\phi(t) \stackrel{\text{def}}{=} \frac{P_0(t)}{\|v\|(t)\ \cdot\ \|i\|(t)} \tag{4}$$

III. Time-Variant Fourier Series

Our time-variant Fourier series representation is based on the so-called Short-Time Fourier Transform (STFT). This means we associate a Fourier series representation with the finite signal segment

$$\{x(s);\ \ t-T\ < s\ \leq t\} \tag{5}$$

where we consider "t" as a parameter, viz.

$$x(s) = \sum_{k=-\infty}^{\infty} X_k(t)e^{jk\omega s}, \quad t-T\ < s\ \leq t \tag{6a}$$

and

$$X_k(t) = \frac{1}{T}\int_{t-T}^{t} x(s)e^{-jk\omega s}ds \tag{6b}$$

where T is the duration of a cycle (or the period in a steady state). We are interested in energy processing systems where this quantity is typically fixed in advance to $1/60$ (or $1/50$). The harmonic coefficient $X_k(t)$ is a function of the parameter t except when $x(\cdot)$ is a periodic signal with period T: in this case $X_k(t)$ is independent of t, and coincides with the standard Fourier coefficient.

The Parseval identity for the Fourier series (6a) can be written in the form

$$\langle\ x\ ,\ y\ \rangle(t) = \sum Y_k^H(t)X_k(t) \tag{7}$$

where we used our definition (2), and where the superscript H denotes complex conjugate (Hermitian) transpose. This means that the space of polyphase signals $x(\cdot)$ on $(t-T, t]$ and the space of their Fourier coefficients $\{X_k(t)\}$ are *isometric* , and can be used interchangeably. We shall carry our subsequent discussion mostly in terms of Fourier coefficients. Notice that since $x(\cdot)$ is real valued, its Fourier coefficients must be conjugate-symmetric, i.e., $X_{-k}(t) = X_k^*(t)$.

IV. Generalized Orthonormal Series

Instead of the Fourier series (6a) we can use some other orthonormal basis of the Hilbert space of square integrable waveforms on $[0,\ T]$, such as the Haar basis [5]. This means that (6a) is replaced by

$$x(s) = \sum_k \chi_k(t)\phi_k(t-s), \qquad t-T\ < s\ \leq t \tag{8a}$$

and

$$\chi_k(t) = \frac{1}{T}\int_{t-T}^{t} x(s)\phi_k^*(t-s)ds \tag{8b}$$

where we assume that for all k, ℓ

$$\frac{1}{T}\int_0^T \phi_k(\tau)\phi_\ell^*(\tau)d\tau = \delta_{k,\ell} \tag{8c}$$

We call $\chi_k(t)$ the time-variant Haar coefficients of $x(t)$. In particular, our Fourier series representation (6a)-(6b) is obtained by setting

$$\phi_k(t) = e^{-jk\omega t} \quad \text{and} \quad X_k(t) = \chi_k(t)e^{-jk\omega t} \tag{9}$$

Because of orthonormality of $\{\phi_k(\cdot)\}$, the Parseval identity (7) still holds, but with the Fourier coefficients replaced by Haar coefficients.

The inner product expression (8b) can be rewritten, via a change of the integration variable, as

$$\chi_k(t) = \frac{1}{T}\int_0^T x(t-\tau)\phi_k^*(\tau)d\tau \tag{10}$$

which identifies $\chi_k(t)$ as the output of a linear time-invariant filter with impulse response $h_k(\tau) = 1/T\phi_k^*(\tau)$. Typically the basis functions $\{\phi_k(\cdot)\}$ are chosen to decompose the frequency content of the signal $x(t)$ into (partially overlapping) frequency bands. For instance, in the Fourier series case (9) $\chi_k(t)$ is centered at the frequency $k\omega$, which motivates the definition of a baseband equivalent $X_k(t) = \chi_k(t)e^{-jk\omega t}$, leading to our definitions (6a)-(6b).

In the case of Haar representation the orthonormal family is doubly indexed, namely $\left\{\phi_{k,\ell}(\tau);\ 2 \leq k < \infty,\ \ 0 \leq \ell \leq 2^{k-1}-1\right\}$, where

$$\phi_{k,\ell}(\tau) = \sqrt{2^{k-1}}\psi(2^{k-1}\frac{\tau}{T} - \ell) \tag{11a}$$

and

$$\psi(x) = \begin{cases} 1 & 0 < x < \frac{1}{2} \\ -1 & \frac{1}{2} < x < 1 \\ 0 & \text{elsewhere} \end{cases} \tag{11b}$$

In addition, the orthonormal basis includes the "DC function"

$$\phi_{0,0}(\tau) = \begin{cases} 1, & 0 < \tau < T \\ 0, & \text{elsewhere} \end{cases} \tag{11c}$$

and the "first Haar harmonic"

$$\phi_{1,0}(\tau) = \psi(\frac{\tau}{T})$$

Notice that for $k \geq 2$ and for all $0 \leq \ell \leq 2^{k-1}-1$, the time-variant Haar coefficient $\chi_{k,\ell}(t)$ is a time-delayed version of $\chi_{k,0}(t)$, viz.,

$$\chi_{k,\ell}(t) = \chi_{k,0}(t - \ell\frac{T}{2^{k-1}}) \tag{12}$$

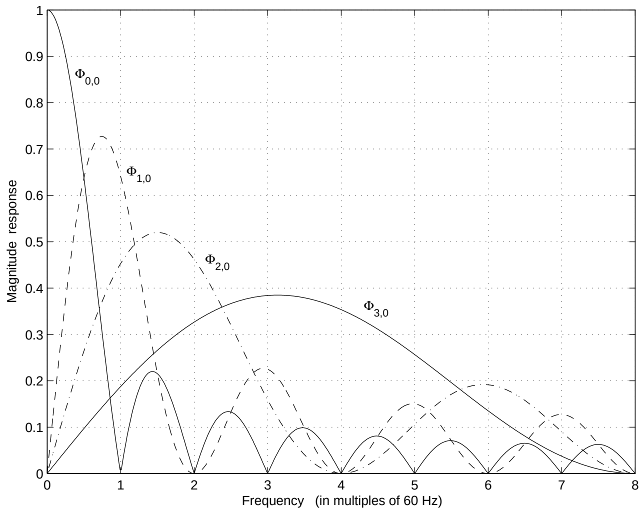

Fig. 1. Haar wavelets in frequency domain (for T=1/60 sec).

Thus, for each value of k we need to evaluate only $\chi_{k,0}(t)$, and we obtain all other $\chi_{k,\ell}(t)$ by applying an appropriate delay. Each $\chi_{k,0}(t)$ represents a different frequency band (see Fig.1).

V. Example

Consider a single-phase RL circuit, driven by the voltage

$$v(t) = \begin{cases} V\sin\omega t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

The resulting current $i(t)$ satisfies the differential equation (for $t \geq 0$) $L^{di}/_{dt} + Ri = v$ which admits the solution

$$i(t) = i_{ss}(t) + i_{tr}(t)$$

with the steady state and transient components of $i(t)$ given by

$$i_{ss}(t) = \frac{V}{R}\ \frac{\sin\omega t - \zeta\cos\omega t}{1+\zeta^2}$$

$$i_{tr}(t) = [i(0) - i_{ss}(0)]\, e^{-\alpha t}$$

where $\alpha \stackrel{\text{def}}{=} {}^R/_L$, $\zeta \stackrel{\text{def}}{=} {}^\omega/_\alpha$. Since the steady-state component $i_{ss}(t)$ is periodic, it consists of a single Fourier harmonic with a time-invariant amplitude and phase. As a result, for $t \geq T$,

$$\frac{1}{T}\int_{t-T}^{t} i_{ss}(s)e^{-jk\omega s}ds = \begin{cases} \frac{V}{2R(j-\zeta)} & k=1 \\ \frac{V}{2R(-j-\zeta)} & k=-1 \\ 0 & \text{else} \end{cases}$$

The transient component, on the other hand, contributes to all harmonics, viz.,

$$\frac{1}{T}\int_{t-T}^{t} i_{tr}(s)e^{-jk\omega s}ds = i_{tr}(0)\frac{e^{\alpha T}-1}{\alpha T}e^{-\alpha t}\frac{e^{-jk\omega t}}{1+jk\zeta}$$

In summary, the time-variant Fourier coefficients of $i(t)$ are (for $t \geq T$)

$$I_k(t) = \begin{cases} i_{tr}(0)\frac{e^{\alpha T}-1}{\alpha T}e^{-\alpha t} & k=0 \\ I_0(t)\frac{e^{-jk\omega t}}{1+jk\zeta} & |k| \neq 1 \\ \frac{V}{2R(j-\zeta)} + I_0(t)\frac{e^{-j\omega t}}{1+j\zeta} & k=1 \\ I_1^*(t) & k=-1 \end{cases}$$

Notice that $|I_k(t)|^2$ is proportional to ${}^{e^{-2\alpha t}}/_{(1+k^2\zeta^2)}$, decaying with time and with increasing harmonic index. In fact, if we choose $R = 0.1\Omega,\ L = 1mH,\ f = 60Hz$, then the only significant harmonics are the zero-th and the first one.

Since the "DC basis function" in the Haar basis is $\phi_{0,0}(\tau) = 1$ for $0 < \tau < 1$, the corresponding coefficient is the same as in the Fourier basis, viz.,

$$\mathcal{I}_{0,0}(t) \equiv I_0(t) = i_{tr}(0)\frac{e^{\alpha T}-1}{\alpha T}e^{-\alpha t}$$

For the remaining Haar coefficients we have

$$\begin{aligned} \mathcal{I}_{k,0}(t) \quad = \quad & -\frac{(e^{\alpha T/2^k}-1)^2}{e^{\alpha T}-1}I_0(t) + \frac{V}{\pi R(1+\zeta^2)} \\ & \left(1-\cos\frac{2\pi}{2^k}\right)\left[\cos\frac{2\pi}{2^k}\left(\cos\omega t + \zeta\sin\omega t\right)\right. \\ & \left. + \sin\frac{2\pi}{2^k}\left(\sin\omega t - \zeta\cos\omega t\right)\right] \end{aligned}$$

In addition (recall (12)), for $k \geq 2$, $\mathcal{I}_{k,\ell}(t) = \mathcal{I}_{k,0}(t - \ell\frac{T}{2^{k-1}})$ for $0 \leq \ell \leq 2^{k-1}-1$. In particular, the first Haar harmonic is

$$\mathcal{I}_{1,0}(t) = -\tanh\frac{\alpha T}{4}I_0(t) - \frac{2V}{\pi R(1+\zeta^2)}(\cos\omega t + \zeta\sin\omega t)$$

Notice that for $k \geq 1$ the Haar coefficients $\mathcal{I}_{k,0}(t)$ consist of both a transient component and a steady state component. This is so because the linear filter used to extract the Haar coefficient $\mathcal{I}_{k,0}(t)$ from the current $i(t)$ has a nonzero gain at the first harmonic frequency ω for all $k \geq 1$ (see Fig.1). As a result, the steady state component $i_{ss}(t)$ "leaks through" this filter, making a contribution to $\mathcal{I}_{k,0}(t)$ for all $k \geq 1$. The magnitude of the steady state component of $\mathcal{I}_{k,0}(t)$ is proportional to $(1 - \cos\frac{2\pi}{2^k})$, which decays exponentially with k, namely, $1 - \cos\frac{2\pi}{2^k} \approx 2\pi^2 \cdot 2^{-2k}$. The magnitude of the transient component of $\mathcal{I}_{k,0}(t)$ is proportional to $e^{-\alpha t}(e^{\alpha T/2^k}-1)^2$, and it also decays with k as 2^{-2k}.

Nevertheless, the "first Haar harmonic" coefficient $\mathcal{I}_{1,0}(t)$ is quite similar to its Fourier counterpart. Since

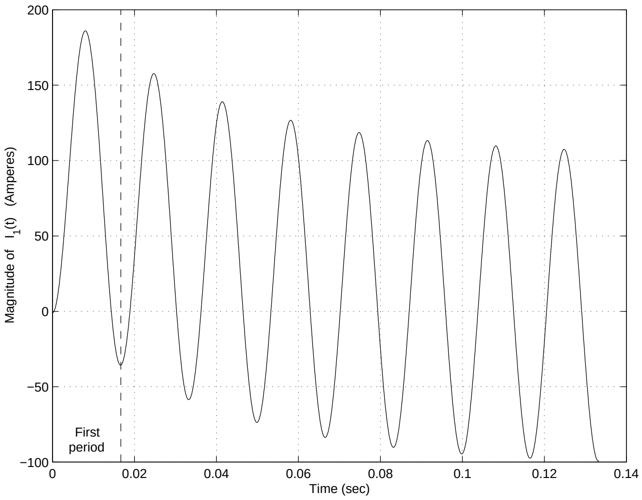

Fig. 2. Current $i(t)$ during the transient.

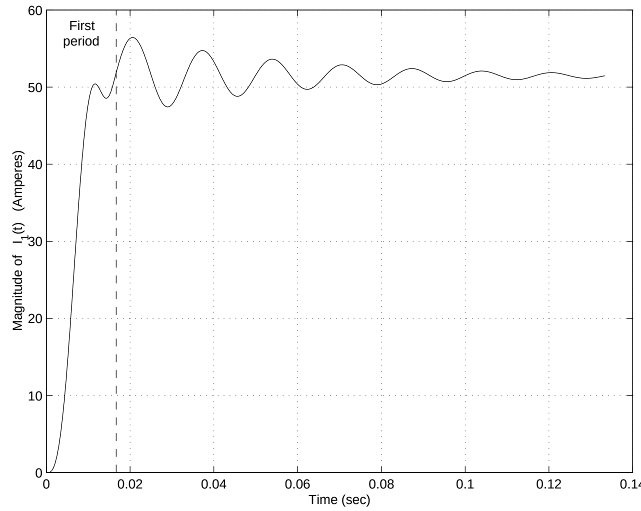

Fig. 3. Magnitude of fundamental harmonic coefficient $I_1(t)$ during the transient.

$\mathcal{I}_{1,0}(t)$ is a passband signal, it should be compared with

$$
\begin{aligned}
2\Re\left\{I_1(t)e^{j\omega t}\right\} &= \frac{1}{1+\zeta^2} I_0(t) \\
&\quad + \frac{V}{R(1+\zeta^2)}\left[\sin\omega t - \zeta\cos\omega t\right]
\end{aligned}
$$

which differs from $\mathcal{I}_{1,0}(t)$ only in terms of amplitude and phase.

In summary, Fourier coefficients provide better frequency localization, at least in steady state, while Haar coefficients provide better time localization, due to the reduced length of the interval used to evaluate them: the Haar coefficient $\mathcal{I}_{k,\ell}(t)$ is evaluated from $i(t)$ values in $\left[t-(\ell+1)\frac{T}{2^{k-1}}, t-\ell\frac{T}{2^{k-1}}\right]$. Since our example involves a smooth gradual transition to steady state, the use of Fourier coefficients appears to be more appropriate.

VI. Time-Variant Powers in the Example

We first display the overall current transient (Fig. 2), where we delineate the first period T with a dashed vertical line. Since $i(t)$ is undefined for $t < 0$, the Fourier coefficients $I_k(t)$ can not be determined for $t < T$. If we choose to define $i(t) = 0$ for $t < 0$, then $I_k(t)$ exhibit an "initialization artifact" for $t \in [0,\ T)$, as is evident from Fig. 3 where we show the time variation of the fundamental harmonic coefficient of the current $I_1(t)$.

The various power components can now be determined from their definitions (1),(3). Thus, using (7),

$$
P_0(t) = \langle\, v\,,\, i\,\rangle(t) = \sum_{k=-\infty}^{\infty} I_k^H(t) V_k(t)
$$

but since in this example $V_k(t) = 0$, except for $|k| = 1$, and $V_1(t) = {}^{V}\!/_{2j} \stackrel{\text{def}}{=} V_1$, we get (for $t \geq T$)

$$
\begin{aligned}
P_0(t) &= V_1 I_1^*(t) + V_1^* I_1(t) = 2\Re\left\{V_1 I_1^*(t)\right\} \\
&= \frac{V^2}{2R(1+\zeta^2)} + V I_0(t)\frac{\zeta\cos\omega t + \sin\omega t}{1+\zeta^2}
\end{aligned}
$$

Of course, as $t \to \infty$, the transient component of the current decays and

$$
\lim_{t\to\infty} P_0(t) = \frac{V^2}{2R(1+\zeta^2)}
$$

Next, $\|v\|^2(t) = {}^{V^2}\!/_2 \equiv 2|V_1|^2$, and

$$
\begin{aligned}
Q^2(t) &= 2|V_1|^2\|i\|^2(t) - \left[2\Re\left\{V_1 I_1^*(t)\right\}\right]^2 \\
&= 2|V_1|^2\left\{\|i\|^2(t) - 2|I_1(t)|^2\right\} \\
&\quad + \left[2\Im\left\{V_1 I_1^*(t)\right\}\right]^2
\end{aligned}
$$

so that $Q^2(t)$ consists of two components:

- $\left[2\Im\left\{V_1 I_1^*(t)\right\}\right]^2$, which is a simple-minded extension from the single phase, single harmonic steady-state expression $2\Im\left\{V_1 I_1^*\right\}$
- $2|V_1|^2\left\{\|i\|^2(t) - 2|I_1(t)|^2\right\}$, which is contributed by the remaining harmonics of the current.

Using the closed form expression for $\|i\|^2(t)$ from [6], viz.,

$$
\|i\|^2(t) = I_0^2(t)\left[\frac{\pi}{\zeta}\coth\frac{\pi}{\zeta} - \frac{2}{1+\zeta^2}\right] + 2|I_1(t)|^2
$$

we obtain

$$
\begin{aligned}
Q^2(t) &= \left[\zeta P_0 + V I_0(t)\frac{\zeta\sin\omega t - \cos\omega t}{1+\zeta^2}\right]^2 \\
&\quad + \frac{V^2}{2} I_0^2(t)\left[\frac{\pi}{\zeta}\coth\frac{\pi}{\zeta} - \frac{2}{1+\zeta^2}\right]
\end{aligned}
$$

As $t \to \infty$, $I_0(t) \to 0$, and $\lim_{t\to\infty} Q(t) = \zeta P_0$, which is the standard steady-state result for this case.

In Fig. 4 we show $P_0(t)$ and $Q(t)$ for our example (with $R = 0.1\Omega$, $L = 1mH$, $f = 60Hz$). As explained earlier,

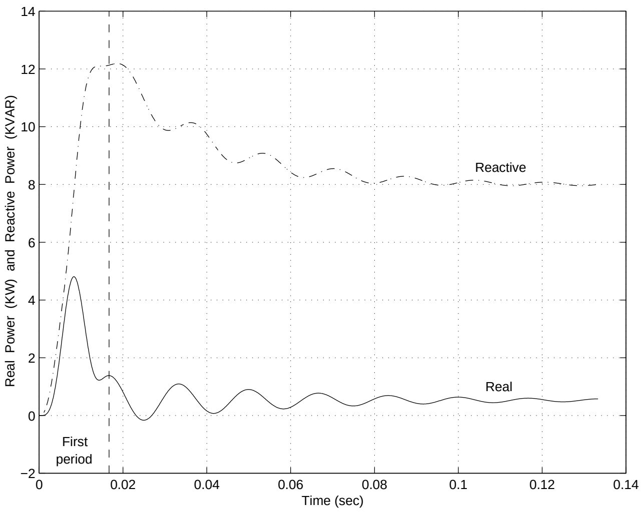

Fig. 4. Transient waveforms of real and reactive power.

our waveforms exhibit a large "initialization artifact" during the first period of the supply waveform (approximately $16ms$), which is caused by considering all waveforms to be zero before $t = 0$ (recall that our closed form expressions hold for $t \geq T = 1/60$).

VII. Dynamics of Local Coefficients

The interpretation (10) of $\chi_k(t)$ as the output of a linear time-invariant filter makes it possible to translate state-space representations of signals into equivalent representations of their expansion coefficients. Thus, if

$$\dot{x}(t) = Ax(t) + Bu(t)$$

then, from (8b)

$$\dot{\chi}_k(t) = A\chi_k(t) + B\upsilon_k(t) \tag{13}$$

where $\upsilon_k(t)$ are the expansion coefficients of the input $u(t)$. It is important to notice that (13) holds for any orthonormal representation. In fact, it is a direct outcome of (10), and does not rely on any specific properties of the waveform $\phi_k(\cdot)$. For example, in the case of Fourier series coefficients (6b), the corresponding equation is of the form

$$\dot{X}_k(t) = (A - jk\omega I)X_k(t) + BU_k(t) \tag{14}$$

where I is the appropriate identity matrix.

When $\upsilon_k(t)$ are known, then $\chi_k(t)$ can be propagated via (13).This requires, of course, to select the initial value $\chi_k(0)$ for each expansion coefficient $\chi_k(t)$. If our choice, say $\hat{\chi}_k(0)$, differs from the true initial value, then the reconstructed $\hat{\chi}_k(t)$ will differ from the true one. We have shown [7] that using a Kalman filter to estimate $\hat{\chi}_k(t)$ results in an optimal reconstruction in the sense of minimizing the error

$$\int_0^t \parallel x(s) - \hat{x}(s) \parallel_2^2 \, ds$$

for all t, where

$$\hat{x}(s) = \sum_k \hat{\chi}_k(t)\phi_k(t - s)$$

is the estimated signal, obtained by using (8a) with the estimated coefficients $\hat{\chi}_k(t)$.

We have also shown that the state space model consisting of (13) together with (8a) is not observable if the only measurement used at time t is $x(t)$ itself [7]. For the same reason, $\chi_k(0)$ can not be recovered from knowledge of $x(0)$ alone. Instead, our Kalman filter uses at every time instant t measurements collected at several instants, i.e., $x(t), x(t-\tau_1), x(t-\tau_2), \ldots, x(t-\tau_M)$ which makes the state space model observable and leads to high quality estimates of $\chi_k(t)$.

VIII. Conclusions

In this paper we proposed a notion of reactive power during transient operation. The definition utilizes the notion of a local Hilbert space and is fully compatible with standard notions in steady state. We presented all concepts in terms of generalized orthonormal coordinates, and explored Fourier and Haar bases in detail. While our illustrative example is quite simple, it also suggests the possibility of on-line control that would aim to compensate reactive power during transients.

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